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ON SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

By Prof. W. W. Johnson, Annapolis, Md.

1. The geometric theory of singular solutions, as it may be called, was given by Cayley in 1872, in the *Messenger of Mathematics*, Vol. II. New Series, pp. 6–12; and further illustrations were given by Cayley in Vol. VI. pp. 23–27, and by Glaisher in Vol. XI. pp. 1–14. The writer employed very much the same method in 1877, in Vol. IV of the *Analyst*, being then unacquainted with Cayley's work. The main points of the theory are as follows: The differential equation being a rational integral equation of the m^{th} degree with respect to $p\,(= dy\,/\,dx)$, the coefficients being one-valued functions of x and y, the integral is in like manner a rational integral equation of the same degree with respect to an arbitrary constant c, having coefficients of the same character. A singular solution satisfies the condition for equal roots in the p-equation; it also satisfies the same condition in the c-equation and represents the envelope of the system of curves represented by that equation. But each of these conditions is also satisfied by other relations between x and y. Namely, if $E = 0$ is the equation of the envelope of the system of curves, then the first condition, or equation in which the p-discriminant is equated to zero, is

$$ECT^2 = 0,$$

where $C = 0$ is the locus of the cusps of the system of curves, and $T = 0$ the *tac-locus*, or locus of points where two curves of the system touch one another. Again, the second condition, in which the c-discriminant is equated to zero, is

$$EC^3N^2 = 0,$$

where E and C have the same meanings as before, and $N = 0$ is the locus of the nodes, or ordinary double-points of the system of curves.

2. This last conclusion Cayley derives from a consideration of the c-discriminant as the locus of the intersections of consecutive curves of the system repre-

sented by the c-equation, showing that, if the variable curve has a permanent node or cusp, there will be two of these intersections which ultimately coincide with each node, and three which coincide with each cusp. If the curve is algebraic and of the m^{th} degree there will remain

$$m^2 - 2\delta - 3\varkappa$$

(which by Plücker's equations is $m + n$, and thus always a positive integer) intersections whose locus is the envelope. There is thus always an envelope real or imaginary, if the integral is algebraic; but as there is not generally a singular solution, it is inferred that a differential equation has not generally an algebraic integral.

3. A consideration of the subject more especially from the point of view of the differential equation, may not be without interest.

The differential equation is a relation between x, y, and p, where if x and y are the rectangular co-ordinates of a point, p is the tangent of the inclination of its motion to the axis of x. We may give to x and y any values we please, and determine m values of p. Thus the equation may be regarded as satisfied by a point having any position, provided only it be moving in one of the directions proper to its position. Consider now a moving point starting from the position (x_0, y_0) and satisfying the differential equation. As it moves, the values of x and y vary, and this in general causes p, as determined by the equation, to vary; so that the point, in order to continue to satisfy the equation, must in general move in a curve. The point may return to its original position describing a closed curve, or it may pass to an infinite distance describing an infinite branch. We have thus a curve such that, if a point move along it in either direction, it satisfies the differential equation; and if we can determine the equation of this curve we shall have a solution of the differential equation. Furthermore, if the solution so determined contains an arbitrary quantity c independent of x and y, the initial point (x_0, y_0) may be any point we please; but to each point (x, y) there must correspond as many values of c as of p, so that the initial value of p may be any one of the m values which are consistent with the values assumed for x and y. If this be the case, we shall have a complete solution of the differential equation in the sense that to every moving point representing simultaneous values of x, y, and p which satisfy the differential equation, there corresponds, if we properly determine c, a curve in which the point may be assumed to be moving. Such a solution is the complete integral.

But, if in the complete solution we are to include the equations of all the curves in which, if the point be moving, it will satisfy the differential equations, then we have still to inquire whether there be any other curves in which the point may be moving. If there be, it is clear that they must at each point coincide in direction with one of the curves included in the complete integral; in

other words, they must be envelopes of the system of curves represented by the complete integral.

4. Since, in general, there must be, corresponding to any point (x, y), as many values of c as of p, the degree of the c-equation, when reduced to its standard form, must be the same as that of the p-equation; but with regard to some special points, the following considerations may be noticed:—

When the equations are of the first degree, the standard forms are for the p-equation

$$Lp + M = 0,$$

and for the c-equation,

$$Pc + Q = 0,$$

where L, M in the first, and P, Q in the second, are one-valued functions of x and y. $L = 0$ is a locus along which p is infinite, that is a locus of points at which the particular integral curves are parallel to the axis of x; but if $x + a$ is a factor of L, $x + a = 0$ is itself a solution. $M = 0$, in like manner, is a locus along which $p = 0$; and if $y + b$ is a factor of M, $y + b = 0$ is a solution. But at the intersections of $L = 0$ and $M = 0$, p is indeterminate. The system of curves is in many cases a pencil of curves passing through these points (and having no other intersections with one another), c being indeterminate at the same points, so that $P = 0$ and $Q = 0$ intersect in these points. But in other cases, the indeterminate value of p indicates that a particular curve of the system has a node at the point in question. As simple illustrations of the two cases, take the equations $xp - y = 0$ and $xp + y = 0$; in both cases the loci in question, $x = 0$ and $y = 0$, are themselves solutions and their intersection is the origin. In the first case the integral is $x - cy = 0$, representing a pencil of straight lines through the origin; in the second the integral is $xy - c = 0$, and the special solutions $x = 0$ and $y = 0$ make up the particular integral corresponding to $c = 0$, namely, $xy = 0$, which has a node at the origin.

5. When the equations are of the second degree, the standard form of the p-equation is

$$Lp^2 + Mp + N = 0$$

where L, M, and N are one-valued functions of x and y. There are generally two values of p at the point (x, y). $L = 0$ is a locus along which one of the values of p is infinite; $N = 0$ is a locus along which one of the values of p is zero. There are generally no points at which p is indeterminate, since at the intersection of $L = 0$ with $N = 0$ we have not generally $M = 0$. The locus of points for which the two values of p are equal, is

$$M^2 - 4LN = 0.$$

The first member of this equation, or p-discriminant, is the square of the differences of the roots of the p-equation. If these roots are rational functions of x

and y, the discriminant is the square of a rational function; the equation can then be decomposed into two equations of the first degree, which give rise to two distinct systems of curves. Through every point (x, y) passes one curve of each system corresponding to the two values of p; but when (x, y) is on the discriminant locus, there is but one value of p, and the two curves, one of each system, touch one another. There is, in this case, no propriety in combining the two integrals into a single expression as Boole does, because there is no determinate manner of pairing together, so to speak, the curves of one system with those of the other, as is done in the assumption that the constant of integration is the same in each system.

6. In the proper, or indecomposable equation of the second degree, the two values of p are the two values of an irrational function of x and y, their difference is a real quantity when the roots are real, and a purely imaginary quantity when the roots are imaginary; thus the discriminant is positive when p has real values, and negative when the values of p are imaginary. The discriminant may break up into factors, and we have to consider the loci on which each of these factors vanish. We have particularly to distinguish between those which occur in the discriminant with an even exponent, so that when (x, y) passes over the locus the discriminant does not change sign, and the values of p remain real; and those which occur with an odd exponent, so that the value of p becomes imaginary as (x, y) passes over the locus.

There is no reason whatever to expect that the values of p when they become equal, should be identical with the value of p for a point moving along the locus of equal roots; that is, that the locus itself should be a solution of the differential equation. This may, however, happen and will constitute a special case to be examined.

7. Imagine now the point (x, y) carrying with it the two intersecting arcs of particular integrals which pass through it, to be moved up to one of the loci of equal roots; and first, in the general case, where the factor which vanishes at the locus occurs in the discriminant in the first (or any odd) degree, and the value of p at the locus is not that corresponding to a point moving along the locus. When the point (x, y) is near the locus, the two arcs passing through it meet the locus at an angle, and do not cross it; and when the point is moved up to the locus they become tangent to one another and form a cusp. The two arcs now belong to the same curve, and there is corresponding to the point (x, y), when on the locus, a single value of c. Hence the factor which vanishes at the locus appears also in the c-discriminant; and, because c becomes imaginary when p does, it therein appears with an odd exponent. In fact, as before mentioned, it appears as a cubic factor; the possession of a cusp is of course a peculiarity of the integral equation, implying the existence of a relation between

the coefficients, but it is not such with respect to the differential equation, just as it is not with respect to the tangential equation of a curve.

In the next place, still supposing the factor which indicates the locus to be of an odd degree, suppose the value of p to be the same as that for a point moving along the locus. In this case, when (x, y) is moved up to the locus the two arcs do not, indeed, cross the locus, but they become tangent to it, and each forms the ordinary continuation of the other. There is, as before, a single value of c corresponding to the point (x, y) when on the locus, and the factor enters the c-discriminant with an odd exponent, generally unity. The locus is an envelope, which, unlike the cusp-locus, does not imply any peculiarity in the integral equation.

In the third place, suppose the factor which vanishes at the locus to be a squared factor, so that the value of p does not become imaginary as we cross the locus, and that the equal values of p differ from the value of p for a point moving in the locus. In this case, the two arcs cross the locus, and when (x, y) is moved up to the locus they simply become tangent to one another, the values of c remaining distinct; we have the tac-locus, the factor not appearing at all in the c-discriminant.

Finally, the locus being still represented by a squared factor, suppose the equal values of p to be identical with the value of p for a point moving in the locus. In this case, the two arcs of particular integrals, instead of crossing the locus, when (x, y) is moved up to it, will coincide with it in direction as in the case of the envelope. But since p is real on both sides of the locus, when (x, y) is moved across the locus, the arcs will reappear on the other side, passing, as it were, bodily across the locus. This implies that the arcs coincide with the locus when (x, y) is upon it. Thus the locus coincides with a particular integral;* namely, an integral which is given doubly in the system, either by a single value of c, or as parts of the integrals given by different values of c. In the first of

* This conclusion, or the equivalent statement that an envelope cannot be represented in the p-discriminant by a squared factor, requires modification when the degree of the p-equation is higher than the second, from the fact that more than two values of p can become equal. To show this, imagine a curve with a point of inflexion at A; and let B and C be neighboring points, one on either side of A, at which the tangents are parallel. Let the system of curves be the result of moving this curve in the direction of the tangents at B and C. These tangents will then be envelopes; each of them separates a region of the plane in which p has at every point three real values, from one in which p has but one real value, the p-equation being, suppose, of the third degree. Now suppose the points B and C to be taken nearer to A, the region in which the three roots are real is narrowed, the two envelopes approaching, until they finally coincide. The envelope is now represented by a squared factor; only one value of p is real on either side of it, but as we cross it the three roots become equal at once. Thus, when $n > 2$, a squared factor of the p-discriminant may represent an envelope which touches the curves of the system at points of inflexion, and at which three values of p become equal. Similar conclusions follow when more than three values of p become equal.

these cases, the factor will appear in the c-discriminant with an even exponent; in the second case the factor will not appear in that discriminant at all.

8. The relation between the several cases enumerated above is well illustrated by means of the orthogonal trajectories of the given system of curves. The differential equation of the orthogonal trajectories, being the result of substituting $-1/p$ for p in the given differential equation, has precisely the same discriminant as the given equation. Now the trajectories will meet an envelope of the given system at right angles; and since the values of p become imaginary in both equations as we cross the locus, the envelope will be a cusp-locus of the trajectories. Conversely, a cusp-locus which is at each point perpendicular to the curves of the given system becomes an envelope of the trajectories; but every other cusp-locus is also a cusp-locus of the trajectories. In like manner, a tac-locus becomes a tac-locus of the trajectories, except when it crosses the touching curves of the system at right angles, in which case it is itself a trajectory, and comes under the fourth of the cases considered above.

9. The consideration of the orthogonal trajectories serves also to illustrate the relation between the cases in which p is at a certain point indeterminate; for such points evidently have the same character in each of two mutually orthogonal systems. Thus, resuming the illustrations given in § 4, $xp - y = 0$ gave us a pencil of lines passing through the origin; the differential equation of the trajectories is $x + py = 0$, which gives $x^2 + y^2 = c^2$, a system of circles in which the origin is an acnode corresponding to $c = 0$. In the other case, $xp + y = 0$, the system represented was the system of hyperbolas $xy - c = 0$, where the origin was a crunode corresponding to $c = 0$; and here the differential equation of the trajectories is $x - py = 0$, which gives $x^2 - y^2 = c^2$, another system of hyperbolas in which the origin is also a crunode.

ON COMPOUND AND REVERSE CURVES.

By Prof. Wm. M. Thornton, University of Virginia, Va.

Curves consisting of a series of circular arcs each tangent to its predecessor are of constant use in the constructive problems of engineering and architecture; as for example for the centre lines of railways, highways, etc.; in the so-called basket-handled arches and in many of the arched forms of the pointed styles of architecture; in machine construction, and so on.

In the following note on these curves primary reference will be made to the location of curves for railways. But the results obtained will be applicable *mutatis mutandis* to every class of these problems.

We consider first the simple curve of a single circular arc. The curvature of these curves is measured most conveniently by the angle at the centre subtended by a chord of unit length. If we put (Fig. 1)

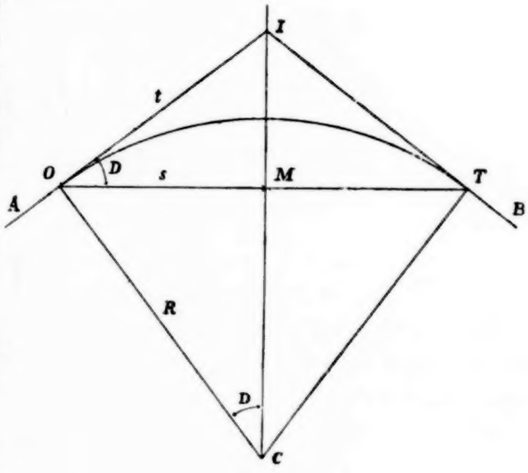


Fig. 1.

$R =$ the radius of the curve,

$\Delta =$ curvature of the curve as above defined,

$s =$ semi-chord of the arc,

$t =$ tangent measured from the point of contact to the intersection,

$D =$ angle between the chord and the tangent, called the deflection,

we shall have $\sin D = \dfrac{s}{R}, \quad \tan D = \dfrac{t}{R};$

or, as $s = \frac{1}{2}$ when $D = \frac{1}{2}\Delta$, $\sin \frac{1}{2}\Delta = \dfrac{1}{2R};$

whence, by eliminating R, we get

$$\sin D = 2s \sin \tfrac{1}{2}\Delta, \quad \tan D = 2t \sin \tfrac{1}{2}\Delta.$$

These are the fundamental relations by the aid of which all problems on simple curves are solved.

If the curvature Δ be given, then two other independent data fix the curve; as AI and BI; AI and O; O and T; and so on.

If the origin O be given, then as before, two other independent data fix the curvature and determine the curve. In this case, however, considerations of con-

venience confine us to integral values of J in minutes; and if necessary, the origin is shifted to conform to such a value. For example, if we have the length of the tangent OI given as 11.23, and the difference of direction of the tangents as $31^\circ\ 12'$, we have

$$D = 15^\circ\ 36', \quad 2t = 22.46;$$

$$\therefore \quad \sin \tfrac{1}{2}\mathrm{J} = \frac{\tan 15^\circ\ 36'}{22.46}$$

$$= [8.09451];$$

$$\therefore \quad \tfrac{1}{2}\mathrm{J} = 42'\ 44''.2, \text{ nearly.}$$

As it would be excessively inconvenient to have to have to set out such an angle repeatedly in the field, we take

$$\tfrac{1}{2}\mathrm{J} = 43';$$

$$\therefore \quad 2t = \frac{\tan 15^\circ\ 36'}{\sin 43'}$$

$$= [1.34874];$$

$$\therefore \quad t = 11.16,$$

and we have to shift O towards I through 0.07.

In the case of the simple curve the angles at O and T are necessarily equal. If these be unequal, we must have recourse to a compound curve of at least two circular arcs. Since the two arcs obey but five conditions,—pass through two points, touch two lines, and touch each other,—it is obvious that there is a sixth remaining condition, which can be arbitrarily imposed upon them.

If A_1J, A_2J (Fig. 2) be the two branches of such a curve, tangent at A_1, A_2 to A_1I, A_2I, and at J to the common tangent T_1T_2, it is easy to see that with the same notations as have been used for simple curves, $T_1 = 2D_1$, $T_2 = 2D_2$, and we have for the difference of azimuths I of the extreme tangents

$$A_1 + A_2 = 2D_1 + 2D_2,$$

which is the fundamental relation connecting the elements of the two arcs of a compound curve.

Since the external angle J of the triangle A_1JA_2 is equal to

$$D_1 + D_2, \quad \text{or} \quad \tfrac{1}{2}(A_1 + A_2),$$

and is therefore constant, the locus of the point J is a circular arc on A_1A_2 with deflection J and radius

$$r = \frac{c}{2 \sin J}$$

where $c = A_1A_2$.

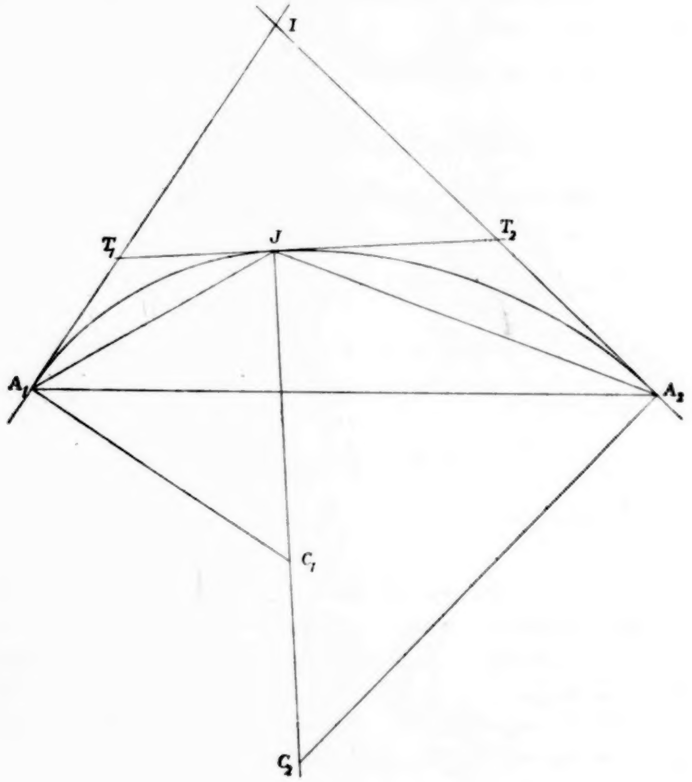


Fig. 2.

Again, since $T_1J = T_1A_1, \quad T_2J = T_2A_2;$

the perimeter of the triangle T_1IT_2 is

$$T_1I + T_2I + T_1T_2 = IA_1 + IA_2,$$

and is therefore constant. Accordingly, the envelope of T_1T_2 is a circular arc tangent to IA_1, IA_2 at points distant $\frac{1}{2}(IA_1 + IA_2)$ from I, whose radius, therefore, is

$$\rho = \frac{l_1 + l_2}{2 \tan J},$$

where $l_1 = IA_1$ and $l_2 = IA_2$.

The point J or T_1T_2 having been selected in accordance with either of these conditions, the two arcs of the compound curve are completely determined.

In fact, if we assign to D_1 (say) any arbitrary value, we have at once $D_2 = J - D_1$, and thence the angles $A_1 - D_1$, $A_2 - D_2$ of the triangle A_1JA_2; whence

$A_1J = 2s_1$ and $A_2J = 2s_2$ are determined, and from these the curvatures of the arcs are computed. In symbols

$$2s_1 = 2r \sin(A_2 - D_2), \quad 2s_2 = 2r \sin(A_1 - D_1);$$

$$\therefore \quad \sin \tfrac{1}{2}J_1 = \frac{\sin D_1}{2r \sin(A_2 - D_2)}, \quad \sin \tfrac{1}{2}J = \frac{\sin D_2}{2r \sin(A_1 - D_1)}.$$

Again, if M_1, M_2 be the points of contact of the arc whose radius is ρ,

$$T_1M_1 = \rho \tan D_1, \quad T_1A_1 = R_1 \tan D_1;$$

whence, since $A_1M_1 = \frac{1}{2}(l_2 - l_1)$

$$2R_1 = 2\rho + \frac{l_1 - l_2}{\tan D_1};$$

and by parity of reasoning,

$$2R_2 = 2\rho - \frac{l_1 - l_2}{\tan D_2}.$$

From either of the two sets of relations the curvatures of the two arcs are easily computed.

For problems of railway location the curvatures thus obtained will usually require a slight adjustment to integral values in minutes, which will require a small shifting of A_1, A_2 towards T_1, T_2, and will leave a short, straight piece between the arcs A_1J_1, A_2J_2.

If it be desired to construct the compound curve with the least possible change of radius, we have from the last relations

$$R_1 - R_2 = \tfrac{1}{2}(l_1 - l_2)(\cot D_1 + \cot D_2)$$

$$= \frac{(l_1 - l_2)\sin J}{\cos(D_1 - D_2) - \cos J}.$$

This difference is obviously least when the denominator is greatest; that is, when

$$D_1 = D_2.$$

If the change of curvature, as measured by the defect from unity of the ratio of the radii,

$$\frac{R_1}{R_2} = \frac{\sin(A_2 - D_2)\sin D_2}{\sin(A_1 - D_1)\sin D_1}$$

$$= \frac{\cos A_2 - \cos(A_2 - 2D_2)}{\cos A_1 - \cos(A_1 - 2D_1)}$$

$$= 1 - \frac{\cos A_1 - \cos A_2}{\cos A_1 - \cos(A_1 - 2D_1)},$$

is to be least, we must have

$$2D_1 = A_1; \quad \therefore \quad 2D_2 = A_2;$$

and hence T_1T_2 parallel to A_1A_2.

If T_1A_1, T_2A_2 lie on opposite sides of the chord A_1A_2, the curvature of the second branch must be reversed. We have then

$$I = A_2 - A_1 = 2D_2 - 2D_1,$$

and therefore

$$J = D_2 - D_1 = \tfrac{1}{2}(A_2 - A_1).$$

The locus of J is the circular arc on A_1A_2 with radius

$$r = \frac{c}{2 \sin J}.$$

Since $JT_1 = IA_1 - IT_1$, $JT_2 = - IA_2 + IT_2$, and therefore

$$T_1T_2 = (T_2I - T_1I) + (A_1I - A_2I),$$

the envelope of T_1T_2 is the circular arc which touches IA_1, A_2I produced at M_1, M_2, distant $\frac{1}{2}(l_1 - l_2)$ from I, the radius therefore being

$$\rho = \frac{l_1 - l_2}{2 \tan J}.$$

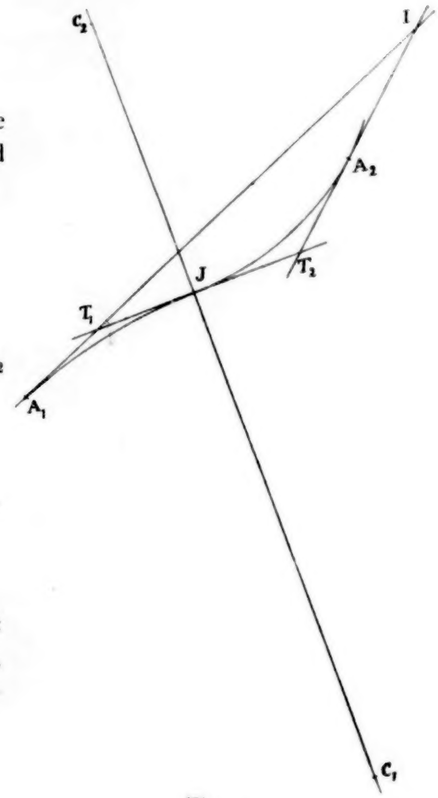


Fig. 3.

As in the previous case, we find

$$\sin \tfrac{1}{2}J_1 = \frac{\sin D_1}{2r \sin (D_2 - A_2)}, \quad \sin \tfrac{1}{2}J_2 = \frac{\sin D_2}{2r \sin (A_1 - D_1)};$$

and

$$2R_1 = \frac{l_1 + l_2}{\tan D_1} - 2\rho, \quad 2R_2 = 2\rho - \frac{l_1 + l_2}{\tan D_2}.$$

The same remarks as to the computation of the curvatures from these formulæ may be repeated here as were made in the former case.

In the special case where the tangents A_1T_1, A_2T_2 are parallel, $A_1 = A_2 = A$, and hence $D_1 = D_2$, $J = 0$, and $r = \infty$. That is, the locus of J is the chord A_1A_2, and hence

$$D_1 = D_2 = A,$$

$$2s_1 + 2s_2 = c.$$

Assuming either s_1 or s_2, the other is known, and we have

$$\sin \tfrac{1}{2}\Delta_1 = \frac{\sin A}{2s_1}, \quad \sin \tfrac{1}{2}\Delta_2 = \frac{\sin A}{2s_2};$$

or

$$R_1 = \frac{s_1}{\sin A}, \qquad R_2 = \frac{s_2}{\sin A}.$$

Conversely $$R_1 + R_2 = \tfrac{1}{2}c \operatorname{cosec} A,$$

a relation from which either of the radii may be computed when the other is assumed. If it is desired that the curves shall be of equal curvature, we have for the common radius

$$R = \tfrac{1}{4}c \operatorname{cosec} A.$$

To illustrate the foregoing relations, we compute the curvatures of the two branches of a compound curve for which

$$A_1 = 18^\circ\, 42', \quad A_2 = 31^\circ\, 26', \quad c = 11.24.$$

Assuming $D_1 = 10^\circ$, we easily find

$$D_2 = 15^\circ\, 4', \quad A_1 - D_1 = 8^\circ\, 42', \quad A_2 - D_2 = 16^\circ\, 22'.$$

We have then

$\log c$	1.05077
$\log \sin J$	9.62703
$\log 2r$	1.42374

$\log \sin (A_2 - D_2)$	9.44992	$\log \sin (A_1 - D_1)$	9.17973
$\log 2s_1$	0.87366	$\log 2s_2$	0.60347
$\log \sin D_1$	9.23967	$\log \sin D_2$	9.41488
$\log \sin \frac{1}{2}\Delta_1$	8.36601	$\log \sin \frac{1}{2}\Delta_2$	8.81141
$\log \tan D_1$	9.24632	$\log \tan D_2$	9.43007
$\log \sin \frac{1}{2}\Delta_{1c}$	8.36678	$\log \sin \frac{1}{2}\Delta_{2c}$	8.81560
$\log 2t_1$	0.88031	$\log 2t_2$	0.61866
$\log 2t_{1c}$	0.87954	$\log 2t_{2c}$	0.61447
$2t_1$	7.591	$2t_2$	4.156
$2t_{1c}$	7.578	$2t_{2c}$	4.116
$t_1 - t_{1c}$	0.0065	$t_2 - t_{2c}$	0.020
$\frac{1}{2}\Delta_1$	$1^\circ 20'-$	$\frac{1}{2}\Delta_2$	$3^\circ 43'-$
$\frac{1}{2}\Delta_{1c}$	$1^\circ 20'$	$\frac{1}{2}\Delta_{2c}$	$3^\circ 45'$

The notations Δ_{1c}, Δ_{2c}, t_{1c}, t_{2c} are used to signify the corrected values of Δ_1, Δ_2, t_1, t_2 when the curvatures are taken in entire minutes. The gap at J will be

$$0.0065 + 0.02 = 0.0265.$$

If we employed the other set of formulæ, we should find

$$l_1 = 7.637, \quad l_2 = 4.695;$$

$$2\rho = 26.367;$$

$$2R_1 = 43.052, \quad 2R_2 = 15.438;$$

$$\log \sin \tfrac{1}{2}\Delta_1 = 8.36601, \quad \log \sin \tfrac{1}{2}\Delta_2 = 8.81141;$$

as above.

The best guide in selecting the curvature of one branch of the compound curve is obtained by introducing into the foregoing formulæ the lengths n_1, n_2 cut off from the normals A_1C_1, A_2C_2 by the bisectrix of A_1IA_2. It is obvious that

$$n_1 = l_1 \cot J, \quad n_2 = l_2 \cot J;$$

whence we find from the foregoing results

$$n_1 - R_1 = \tfrac{1}{2}(n_2 - n_1)\frac{\cot D_1 - \cot J}{\cot J} = \frac{n_2 - n_1}{2\cos J}\cdot\frac{\sin D_2}{\sin D_1},$$

$$R_2 - n_2 = \tfrac{1}{2}(n_2 - n_1)\frac{\cot D_2 - \cot J}{\cot J} = \frac{n_2 - n_1}{2\cos J}\cdot\frac{\sin D_1}{\sin D_2},$$

and therefore $$(n_1 - R_1)(R_2 - n_2) = \left(\frac{n_2 - n_1}{2\cos J}\right)^2,$$

which is the desired relation. It serves to determine one radius as soon as the other is given; and shows that if $l_2 > l_1$ (as can always be assumed), then we must have $R_1 < n_1$, $R_2 > n_2$.

Conversely, R_1, R_2 having been determined, we have

$$\cot D_1 = \cot J\,\frac{n_2 + n_1 - 2R_1}{n_2 - n_1} = \frac{n_2 + n_1 - 2R_1}{l_2 - l_1},$$

$$\cot D_2 = \cot J\,\frac{2R_2 - n_2 - n_1}{n_2 - n_1} = \frac{2R_1 - n_2 - n_1}{l_2 - l_1};$$

relations that determine the deflections from the radii.

For the reverse curves we obtain the like relations,

$$n_1 + R_1 = \frac{n_2 + n_1}{2\cos J}\,\frac{\sin D_2}{\sin D_1},$$

$$n_2 + R_2 = \frac{n_2 + n_1}{2\cos J}\,\frac{\sin D_1}{\sin D_2},$$

$$(n_1 + R_1)(n_2 + R_2) = \left(\frac{n_1 + n_2}{2\cos J}\right)^2,$$

$$\cot D_1 = \frac{n_2 - n_1 - 2R_1}{l_2 + l_1}, \quad \cot D_2 = \frac{n_2 - n_1 + 2R_2}{l_2 + l_1}.$$

It would be easy to deduce these relations from the following theorem analogous to that of page 41 : —

The common normal C_1C_2 envelopes a circle concentric with the circle (ρ) and tangent to the end normals A_1C_1, A_2C_2. The centre of both circles is the intersection point of the bisectrix of I with the bisectrix of the angle N between the normals. The radius of the new circle is $\frac{1}{2}(l_2 - l_2)$.

An interesting problem in railway engineering is furnished by curved turnouts from a curved main line. In this case the inner line of the outer rail of the turnout meets that of the inner rail of the main line at F, the point of the frog. Putting G for the gauge, measured between the inner lines of the rails, and F for the frog angle, we have in the triangle CC_0F

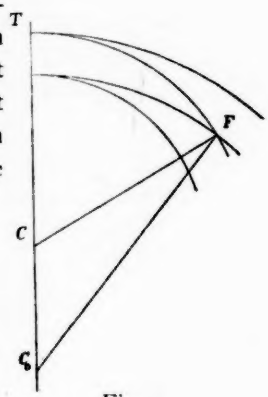


Fig. 4.

$$C = \pi - 2D, \quad C_0 = 2D_0, \quad F = F;$$

$$C_0F = R_0 - \tfrac{1}{2}G, \quad CF = R + \tfrac{1}{2}G, \quad CC_0 = R_0 - R;$$

and by the usual formulæ of trigonometry,

$$\tan D = \frac{G}{2R}\cot\tfrac{1}{2}F,$$

$$\tan D_0 = \frac{G}{2R_0}\cot\tfrac{1}{2}F,$$

$$D = D_0 + \tfrac{1}{2}F.$$

These formulæ enable us to compute all the elements of the turnout when F is given. If R or Δ is given, we use the derived formula

$$\tan^2\tfrac{1}{2}F = G(\sin\tfrac{1}{2}\Delta - \sin\tfrac{1}{2}\Delta_0 - \sin\tfrac{1}{2}\Delta\sin\tfrac{1}{2}\Delta_0)$$

to find F; whence the other elements as before.

If the main track is straight, $\Delta_0 = 0$, and

$$\tan^2\tfrac{1}{2}F = G\sin\tfrac{1}{2}\Delta;$$

from which relation either F or Δ is known when the other is given.

If the turnout is on the convex side of the main track, it is only necessary to change the sign of R_0 or Δ_0.

INTEGRATION OF RICCATI'S EQUATION.

By MR. LEVI W. MEECH, Norwich, Conn.

Since the publication of my paper on the Integration of Riccati's Equation * it has been found that the series may take the form of a *definite integral.* A practical application occurs in the problem of mechanics, in tracing the path of a projectile in a resisting medium with friction as the square of the velocity. By a well-known integral, and by multiplying both numerator and denominator of the common binomial series by equal factors, which does not alter the value, we have

$$\int_0^\infty y^m \varepsilon^{-y}\, dy = 1 \,.\, 2 \,.\, 3 \,.\, \ldots \,.\, m = m\,!,$$

$$(1+b)^m = \frac{m\,!}{m\,!} + \frac{m\,!\, b}{(m-1)\,!} + \frac{m\,!\, b^2}{1 \,.\, 2\,(m-2)\,!} + \frac{m\,!\, b^3}{1 \,.\, 2 \,.\, 3\,(m-3)\,!} + \ldots$$

Hence in the last formula (18),† where b and c have *like* signs, by making

$$2q'z = \frac{4}{n}\sqrt{(bcx^n)} = h$$

and

$$\tfrac{1}{2}h + C' = t,$$

if

$$\frac{du}{dx} + bu^2 = cx^m,$$

$$N = \int_0^\infty \frac{y^i \varepsilon^{-y}}{i\,!}\left\{\left(1+\frac{y}{h}\right)^i \varepsilon^{-t} + \left(1-\frac{y}{h}\right)^i \varepsilon^{+t}\right\} dy,$$

$$\varDelta = \int_0^\infty \frac{y^{i-1} \varepsilon^{-y}}{(i-1)\,!}\left\{\left(1+\frac{y}{h}\right)^{i-1} \varepsilon^{-t} - \left(1-\frac{y}{h}\right)^{i-1} \varepsilon^{+t}\right\} dy,$$

$$u = -\frac{N}{\varDelta}\cdot\sqrt{\left(\frac{cx^m}{b}\right)}. \tag{19}$$

By changing the variable from y to θ, making

$$y = \tan\theta, \quad dy = \frac{d\theta}{\cos^2\theta},$$

to facilitate quadrature, the limits of the definite integral become $\frac{1}{2}\pi$ and 0. In the common process, the quadrant is divided into three or more equal parts, and N, $\cos^2\theta$, and $\varDelta$ are computed for $\theta = 15°, 45°, 75°$, or for the middle of each

* *Annals of Mathematics*, Vol. I. pp. 97–103.
† Ibid, p. 102.

part. Since $i!$ in N, divided by $(i-1)!$ in J, gives simply i, the example where i is not an integer will occasion no difficulty, and the quadrature converges rapidly.

When b and c have *unlike* signs, the factor $\sqrt{(-1)}$ occurring in h, and in $\sqrt{\left(\frac{cx^m}{b}\right)}$, will evidently change the sum and difference of ε^{-t} and ε^{t} into cos t and sin t, requiring a separate formula. It should be noted that on p. 102 a few errors in algebraic signs may be corrected from the preceding page, or from the definite integral (19). If, in verification of J, we add 1 to i, the two parts of J evidently become identical with N, except the connecting sign. Also when $i=0$, and $m=0$, the result of (18) or (19) has the same value, with only a change in form, as equation (2), where the arbitrary constant

$$C=-\varepsilon^{-2c'}, \quad \text{and} \quad u=-\sqrt{\frac{c}{b}}\cdot\frac{\varepsilon^{-t}+\varepsilon^{t}}{\varepsilon^{-t}-\varepsilon^{t}}\cdot\frac{\varepsilon^{-t}}{\varepsilon^{-t}}.$$

In integrating by continued fractions when i is an integer, the formulæ (20) and (21) are here materially simplified from p. 99 of Vol. I, compared with Boole's *Differential Equations*. The first of the two forms of the continued fraction is found from Riccati's Equation by assuming and substituting successively from $\frac{du}{dx}+bx^2=cx^m$,

$$m+2=n, \quad \sqrt{(cb)}=k, \quad u=\frac{ky}{bx};$$

$$y=\frac{1}{k}+\frac{x^n}{y_1},$$

$$y_1=\frac{1+n}{k}+\frac{x^n}{y_2},$$

$$y_2=\frac{1+2n}{k}+\frac{x^n}{y_3},$$

$$\cdots$$

$$y_{i-1}=\frac{1+(i-1)n}{k}+\frac{x^n}{y_i},$$

$$\frac{xdy_i}{dx}-(1+in)y_i+ky_i^2=kx^n.$$

When i is a positive integer, $1+in=\frac{1}{2}n$, and the last equation gives the exact value of y_i through (2) or (3), by first making $x^n=x_1^2$, and $y_i=zx_1$; whence

$$\frac{dz}{dx_1}+\frac{2}{n}kz^2=\frac{2}{n}k.$$

Then writing out the two-fold values of $y, y_1, \ldots y_i$, reducing, and making $k^2x^n = t$, we have

$$bx \,.\, u = 1 + \frac{t}{1+n+} \frac{t}{1+2n+} \frac{t}{1+3n+} \cdots \frac{t}{1+(i-2)\,n+} \frac{t}{-\frac{1}{2}n+} \frac{t}{ky_i}. \tag{20}$$

This gives u for the series (5), where $n = \dfrac{-2}{2i-1}$. Again, for the series (6), or $n = \dfrac{2}{2i+1}$,

$$bx \,.\, u = \frac{t}{n-1+} \frac{t}{2n-1+} \frac{t}{3n-1+} \cdots \frac{t}{(i-1)\,n-1+} \frac{t}{-\frac{1}{2}n+} \frac{t}{ky_i}, \tag{21}$$

where
$$ky_i = \sqrt{t} \,.\, \frac{C\varepsilon^{\frac{4\sqrt{t}}{n}} + 1}{C\varepsilon^{\frac{4\sqrt{t}}{n}} - 1},$$

$$= \sqrt{-t} \,.\, \cot\left(C + \frac{2\sqrt{-t}}{n}\right),$$

according as b and c have like or unlike signs. The simplified formula (21) differs apparently from the result of Prof. Boole, but the sum and the number of terms in both, is the same. For example, when $m + 2 = n = \frac{2}{9}$, or $i = 4$, the formula (21) becomes

$$bx \,.\, u = \frac{t}{-\frac{7}{9}+} \frac{t}{-\frac{5}{9}+} \frac{t}{-\frac{3}{9}+} \frac{t}{-\frac{1}{9}+} \frac{t}{ky_i}.$$

When i is not an integer, let i' denote its value. When i' differs but little from an integer, we may suppose the series $1+n$, $1+2n, \ldots$ running forward from the beginning, and another series $\ldots, -\frac{3}{2}n, -\frac{1}{2}n$, running back from the known integral ky_i, to intersect near $-\frac{1}{2}n$. A sufficient approximation may often be obtained by substituting half the sum of $1+(i-2)\,n$ and $-\frac{1}{2}n$ in the place of $-\frac{1}{2}n$ in the last denominator but one of (20); or half the sum of $in - 1$ and $-\frac{1}{2}n$ in place of $-\frac{1}{2}n$ in (21), without other change of the formulæ. For other values of i' the possibility of connecting the two series is offered by some other less simple method of interpolation.

ON THE CHORD COMMON TO A PARABOLA AND THE CIRCLE OF CURVATURE AT ANY POINT.

By Prof. R. H. Graves, Chapel Hill, N. C.

It is known that if a circle meet a parabola in four points, the sum of the distances of the points on one side of the axis from it is equal to the sum of the distances of the points on the other side from it. If three of the points are coincident, the circle becomes the circle of curvature, and the distance of the three coincident points (P) from the axis is one-third of that of the fourth point from the axis.

Hence the common chord of the circle and parabola is divided by the axis in the ratio 1 : 3. But the shorter segment of the chord is equal to the tangent at P, since they are equally inclined to the axis. Therefore the chord is equal to four times the tangent. Let $y^2 = 4ax$ be the equation to the parabola, and (x', y') the co-ordinates of P. Then

$$y - y' = -\frac{2a}{y'}(x - x'), \quad \text{or} \quad yy' + 2ax - \tfrac{3}{2}y'^2 = 0,$$

is the equation to the chord.

Differentiating with respect to y', $y = 3y'$; hence $y^2 = -12ax$ is the envelope of the chord. Also, from relation $y = 3y'$, it follows that the longer segment of the chord is equal to the corresponding tangent of the parabola, $y^2 = -12ax$.

The point P, and the point where the chord prolonged touches $y^2 = -12ax$, are harmonic conjugates with respect to the points where it meets the axis and the tangent at the common vertex of the parabolas.

The tangent at the end of the *latus rectum* of $y^2 = -12ax$ is normal to $y^2 = 4ax$ at the end of its *latus rectum*, and therefore touches its evolute. The chord is then a diameter of the circle of curvature, and is bisected by its point of contact with the evolute.

Hence the radius of curvature = twice the normal = $4a\sqrt{2}$, which agrees with a known result.

GENERALIZATION OF EXERCISE 97.

By Marcus Baker, Washington, D. C.

Problem.—In the triangle ABC two lines drawn from C intersect the side AB. Given the angle C, the angle between the lines, and the ratios between the segments of AB; to find the remaining angles of the figure.

Solution.—1. Let M, M' be the points of intersection, such that

$$AM : M'B : AB = m : m' : 1.$$

Draw parallels to BC, intersecting AC at N and N'; whence

$$\frac{M'N'}{MN} \cdot \frac{CN}{CN'} = \frac{1-m'}{m} \cdot \frac{1-m}{m'}.$$

From the triangles CMN and $CM'N'$ we have

$$\frac{MN}{CN} = \frac{\sin NCM}{\sin CMN} = \frac{\sin ACM}{\sin MCB},$$

and
$$\frac{M'N'}{CN'} = \frac{\sin N'CM'}{\sin CM'N'} = \frac{\sin ACM'}{\sin M'CB};$$

whence
$$\frac{\sin(\alpha+\varphi)}{\sin[C-(\alpha+\varphi)]} \cdot \frac{\sin(C-\varphi)}{\sin\varphi} = k, \tag{1}$$

where $\varphi = \angle ACM$, $\alpha = \angle MCM'$, and $k = \dfrac{(1-m)(1-m')}{mm'}$.

Clearing of fractions, dividing by $[\cos\alpha\cos C - k\cos(C-\alpha)]\cos^2\varphi$, and transposing, we have

$$\tan^2\varphi - \frac{(1-k)\sin(C-\alpha)}{\cos\alpha\cos C - k\cos(C-\alpha)}\tan\varphi = \frac{\sin\alpha\sin C}{\cos\alpha\cos C - k\cos(C-\alpha)}, \tag{2}$$

from which to find φ.

2. Equation (2) may also be found in the following manner:—

If we designate the sides, as usual, by a, b, c, and put $MC = x$, $M'C = y$, $M'CB = \varphi'$, whence $\varphi + \alpha + \varphi' = C$; then from a consideration of equivalent areas,

$$ab\sin C = \frac{xy\sin\alpha}{1-m-m'} = \frac{bx\sin\varphi}{m} = \frac{ay\sin\varphi'}{m'};$$

whence
$$\frac{abxy\sin C\sin\alpha}{1-m-m'} = \frac{abxy\sin\varphi\sin\varphi'}{mm'},$$

or
$$\sin\varphi\sin\varphi' = \frac{mm'}{1-m-m'}\sin C\sin\alpha. \tag{3}$$

Also, by (1) $\sin\varphi \sin\varphi' = \dfrac{mm'}{(1-m)(1-m')} \sin(a+\varphi)\sin(C-\varphi),$ (4)

or combining (3) and (4),

$$\sin(a+\varphi)\sin(C-\varphi) = \frac{(1-m)(1-m')}{1-m-m'}\sin C \sin a; \tag{5}$$

which by expansion and reduction becomes (2) as already found.

φ' can, of course, be found in a manner entirely analogous; or φ being known, it can be obtained from (3) and checked by the relation $\varphi' = C - (a+\varphi)$.

3. To find A and B we have

$$\frac{\sin\varphi}{\sin A} = \frac{mc}{MC} \quad \text{and} \quad \frac{\sin(C-\varphi)}{\sin B} = \frac{(1-m)c}{MC};$$

whence $\dfrac{\sin B}{\sin A} = \dfrac{\sin(C+A)}{\sin A} = \sin C \cot A + \cos C = \dfrac{m}{1-m}\cdot\dfrac{\sin(C-\varphi)}{\sin\varphi}$;

$$\therefore \quad \tan A = \frac{\sin C}{\dfrac{m}{1-m}\cdot\dfrac{\sin(C-\varphi)}{\sin\varphi} - \cos C},$$

and similarly,
$$\tan B = \frac{\sin C}{\dfrac{1-m}{m}\cdot\dfrac{\sin\varphi}{\sin(C-\varphi)} - \cos C}. \tag{6}$$

4. As φ is obtained by means of a quadratic equation, the problem may be solved by a geometric construction. The following is such a construction: —

Upon any arbitrary line AB construct a circular segment containing the angle C; upon a segment MM' of this line construct another circular segment containing the angle a; the intersections of these segments in C and C' determine two triangles, whose angles A and B satisfy the required conditions. The proof is obvious, as is also the fact that there are in general two solutions.

This solution is due to Mr. James Main, of Washington, D. C., who solves this problem by co-ordinate geometry,* deriving equations to the two circles above drawn.

5. In the special case of a right-angled triangle, $C = \frac{1}{2}\pi$, and equations (2) and (6) become

$$\tan^2\varphi - \frac{1-m-m'}{(1-m)(1-m')}\cot a \tan\varphi = -\frac{mm'}{(1-m)(1-m')} \tag{2'}$$

and
$$\tan A = \frac{1-m}{m}\tan\varphi. \tag{6'}$$

*[See ANNALS OF MATHEMATICS Vol. II. p. 142.—O. S.]

If we substitute in (2′) the value of tan φ from (6′) and reduce, we have

$$\tan^2 A - \frac{1 - m - m'}{m(1 - m')} \cot \alpha \tan A = - \frac{m'(1 - m)}{m(1 - m')},$$

from which we may obtain A directly from the data without calculating the auxiliary φ. Also (3), (5), and (1) become

$$\sin \varphi \sin \varphi' = \frac{mm'}{1 - m - m'} \sin \alpha, \tag{3'}$$

$$\cos \varphi \cos \varphi' = \frac{(1 - m)(1 - m')}{1 - m - m'} \sin \alpha, \tag{5'}$$

$$\tan \varphi \tan \varphi' = \frac{mm'}{(1 - m)(1 - m')}. \tag{1'}$$

6. (6′) may also be derived in the following manner: —

If M_0 be the middle point of AB, N_0 the middle point of AC, and O the intersection of CM and M_0N_0; then, as may be easily seen,

$$\frac{M_0N_0}{ON_0} = \frac{\tan A}{\tan \varphi} = \frac{M_0N_0}{MN} \frac{MN}{ON_0} = \frac{1 - m}{m},$$

or

$$\tan A = \frac{1 - m}{m} \tan \varphi,$$

and similarly

$$\tan B = \frac{1 - m'}{m'} \tan \varphi'. \tag{6'}$$

7. If the middle point of MN' coincides with M_0, $m = m'$; whence

$$\sin \varphi \sin \varphi' = \frac{m^2}{1 - 2m} \sin \alpha, \tag{3''}$$

$$\cos \varphi \cos \varphi' = \frac{(1 - m)^2}{1 - 2m} \sin \alpha, \tag{5''}$$

$$\tan \varphi \tan \varphi' = \frac{m^2}{(1 - m)^2}. \tag{1''}$$

Also, since $\tan B = \cot A$, (6′) gives

$$\tan A + \cot A = \frac{2}{\sin 2A} = \frac{1 - m}{m} (\tan \varphi + \tan \varphi').$$

But

$$\tan \varphi + \tan \varphi' = \frac{\sin(\varphi + \varphi')}{\cos \varphi \cos \varphi'} = \frac{1 - m - m'}{(1 - m)(1 - m')} \cot \alpha;$$

whence

$$\sin 2A = \frac{2m(1 - m')}{1 - m - m'} \tan \alpha, \tag{8}$$

by means of which A may be calculated without the use of auxiliaries.

CARR'S SYNOPSIS.*

To any one interested in mathematics it is valuable not only to possess as many works as possible, each of which is a standard authority on a single subject, but also at least one work which can be referred to because it contains something on nearly every subject. Such a work may be in the form of an encyclopedia arranged alphabetically, or of a compendium arranged topically. Of the latter class *Carr's Synopsis* is an excellent example.

No general epitome of results can, of course, reach that perfection of detail which can be given to a treatise on a single subject, and such perfection is not expected. It is only surprising that Mr. Carr has succeeded in this respect so well.

The book is called a synopsis of elementary results, but it is more than that. Besides chapters on subjects which every one would call elementary, there are pretty full treatises on Differential and Integral Calculus, Calculus of Variations, Differential Equations, Calculus of Finite Differences, and Analytical Conics in Trilinear Co-ordinates. Even the portions devoted to so-called elementary subjects contain much more than the merely fundamental principles.

There are in all nearly one thousand octavo pages, containing more than six thousand separate propositions; but this gives no just estimate of what the work contains, so carefully has the author condensed it in every part.

The topics are carefully arranged and each proposition is numbered, the more important portions being printed in larger type than the remainder. The arrangement, the numbering, and the beautifully clear and heavy type in which the more important portions are printed, added to the detailed tables of contents at the beginning and the equally detailed index at the end make it a work of only a moment to turn to any desired subject.

Besides numerous geometrical figures in the body of the work, twenty plates, containing nearly two hundred figures, are added at the end.

A welcome feature of the index is the index which is joined with it, of the papers on pure mathematics which have appeared during the present century in thirty-two of the leading journals and series of society transactions.

*A Synopsis of Elementary Results in Pure Mathematics. By G. S. Carr. London: Francis Hodgson.

SOLUTIONS OF EXERCISES.

62 and 63

FIND the radius of a spherical dome whose $\frac{\text{volume}}{\text{surface}}$ is 2000% and whose altitude is ⅔ of the radius.

SOLUTION.

If R be the radius of the sphere and $H = \frac{2}{3}R$, the height of the dome, we have for its surface,

$$S = 2\pi RH = \tfrac{4}{3}\pi R^2;$$

and for its volume,

$$V = \tfrac{2}{3}RS - \tfrac{1}{3}\pi(R-H)[R^2 - (R-H)^2] = \tfrac{28}{81}\pi R^3.$$

Hence in (62), $$R = \sqrt[3]{\left(\frac{81000}{14\pi}\right)};$$

and in (63), $$R = \sqrt{\left(\frac{1500}{\pi}\right)}.$$ [*O. L. Mathiot.*]

87

FIND the angle between the axes that $x^2 + xy + y^2 = 0$ may be at right angles.

SOLUTION.

Solving with reference to y, we find that $y = -(\frac{1}{2} \pm \frac{1}{2}i\sqrt{3})x$. The condition that two lines shall be at right angles is, that

$$1 + (m + m')\cos\omega + mm' = 0.$$

These values for m and m', when substituted in the equation of condition, give $\cos\omega = 2$, or the axes are imaginary. The angle between the axes must be $\cos^{-1}2$. If the axes are real and orthogonal, the angle between the lines is $\tan^{-1}\left(\frac{3^{\frac{1}{2}}}{\sqrt{2}}\right)$.

[*Cooper D. Schmitt; R. H. Graves.*]

88

An ellipse referred to equi-conjugate diameters inclined at ω, has for equation $x^2 + y^2 = c^2$. Find the equation with reference to the same axes to the locus of the intersection point of orthogonal tangents.

SOLUTION.

The equation to the pair of tangents drawn from the point (x', y') is,

$$(x^2 + y^2 - c^2)(x'^2 + y'^2 - c^2) = (xx' + yy' - c^2)^2.$$

The condition that these lines should be at right angles gives

$$x'^2 + y'^2 + 2x'y' \cos \omega = 2c^2;$$

$\therefore$ the required equation is $x^2 + y^2 + 2xy \cos \omega = 2c^2$.

Or solve thus: —

The distance from the origin to the intersection point is equal to

$$\sqrt{(a^2 + b^2)} = \sqrt{(2c^2)};$$

$\therefore$ the required equation is $x^2 + y^2 + 2xy \cos \omega = 2c^2$.

[*R. H. Graves.*]

92

FROM the point of contact of two equal circles α, β points A, B move on their circumferences with equal velocities in opposite directions. Find the motion of B relative to A.

SOLUTION.

Suppose the motion of the point A to be compensated by an equivalent motion of the plane of the two circles in the opposite direction. This motion of the plane is equal to that of the point B, and the motion of B is doubled. Hence the motion of B relative to A is the same as if A were to remain stationary, and B to describe a circle, whose diameter is twice that of either of the two equal circles.

The nature of the central force governing such motion may be determined as follows: —

The differential equation of the orbit pertaining to any attractive central force is

$$F = c^2 s^2 \left(\frac{d^2 s}{d\theta^2} + s \right), \tag{1}$$

in which c is some constant, s the reciprocal of the radius vector ($s = 1 / \rho$), and θ the variable angle in polar co-ordinates, the pole being taken at the centre of force.

The polar equation of a circle, the pole being a point in the circumference, is

$$\rho = 2R \cos \theta,$$

from which

$$s = \frac{1}{2R} \sec \theta. \tag{2}$$

Differentiating (2),

$$\frac{ds}{d\theta} = \frac{1}{2R} \tan \theta \sec \theta,$$

and

$$\frac{d^2 s}{d\theta^2} = \frac{1}{2R} (\sec^3 \theta + \tan^2 \theta \sec \theta). \tag{3}$$

Substituting the values given in equations (2) and (3), in equation (1), and reducing,

$$F = 8R^2c^2s^5 = 8R^2c^2 / \rho^5,$$

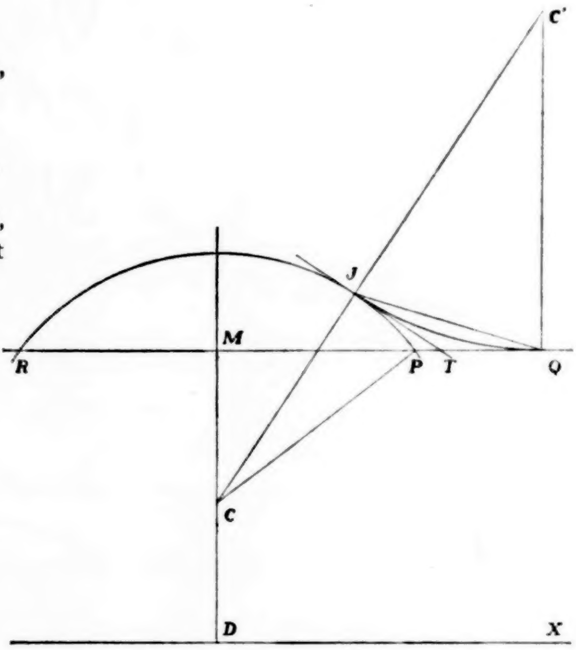
which shows that the force varies inversely as the fifth power of the distance. If the two particles, A and B, have equal masses, their motion, if governed by this law, will be that specified in the problem. [*L. G. Weld.*]

[Solved also by Professor Thornton.]

96

GIVEN on the ground a circular curve of known radius intersecting a given straight line at a given point, and inclined to it at that point at a given angle; it is required to determine the radius of a second circular arc which shall be tangent both to the given curve and to the given line at another given point. [*Calvin Whiteley.*]

SOLUTION I.



Let RJP be the given curve,
PQ the given line,
JQ the required curve,
C, C' the centres,
R, R' the radii,
and $PCM = 2A$ the given angle.

Put $JCP = 2D$, $JC'Q = 2D'$, and draw the common tangent JT. In the triangle JPQ

$$PQ = a,$$
$$PJQ = D + D',$$
$$JP = 2R \sin D,$$
$$JQP = D' = A - D,$$
$$JQ = 2R' \sin D',$$
$$JPQ = \pi - A - D';$$

since $A = D + D'$.

Accordingly

$$a : 2R \sin D : 2R' \sin D' = \sin A : \sin (A - D) : \sin (A + D'),$$

or

$$\frac{2R}{a} = \cot D - \cot A,$$

$$\frac{2R'}{a} = \cot D' + \cot A.$$

The first equation determines D; then $D' (= A - D)$ is known; and the second equation determines R'. [*Frank Muller.*]

SOLUTION II.

C' is on a parabola, $$y^2 = 4dx,$$
whose focus is C, and directrix DX, parallel to PQ, at a distance $MD = R$. We have at once $2d = CD$, or

$$\begin{aligned} d &= R \sin^2 A, \\ y &= \tfrac{1}{2}(a + b) \\ &= a + R \sin 2A, \\ R' &= x + d - R \\ &= x - R \cos^2 A \\ &= \frac{y^2}{4d} - R \cos^2 A \\ &= \frac{y^2 - R^2 \sin^2 2A}{4d}; \end{aligned}$$

whence $$R' = \frac{a^2 + 2aR \sin 2A}{4R \sin^2 A}.$$ [*S. M. Barton.*]

SOLUTION III.

$$CC'^2 = MQ^2 + (CM + QC')^2;$$
$$\therefore \quad (R + R')^2 = (a + R \sin 2A)^2 + (R' + R \cos 2A)^2,$$
whence by an easy reduction,
$$R' = \frac{a^2 + 2aR \sin 2A}{4R \sin^2 A}.$$ [*J. E. Hendricks.*]

SOLUTION IV.

With the same notations, let
$$QP = a, \quad QR = a + 2R \sin 2A = b, \quad CM = R \cos 2A = c;$$
then if $TJ = TQ = t$,
$$t^2 = (a - t)(b - t),$$
$$\therefore \quad t = \frac{ab}{a + b}.$$
From this result we compute
$$TCM = \tan^{-1} \frac{a + b - 2t}{c}, \quad TCJ = \tan^{-1} \frac{t}{R};$$
whence $$2D' = TCM - TCJ,$$
and $$R' = t \cot D'.$$ [*Calvin Whiteley.*]

98

Professor Hall calls attention to the fact that the reduction of

$$\begin{vmatrix} 1 & \cos p_1 & \sin p_1 \\ 1 & \cos p_2 & \sin p_2 \\ 1 & \cos p_3 & \sin p_3 \end{vmatrix}$$

required in the solution of the first part of this exercise, is to be found in Gauss's *Theoria Motus*, Art. 82.

Since the exercise was published, the proposer has found that the results are already given in Salmon's *Conic Sections*, Art. 231.*

100

THE points O, O' defined by the equations in trilinears

$$a\alpha : b\beta : c\gamma = c^2a^2 : a^2b^2 : b^2c^2,$$

$$a\alpha : b\beta : c\gamma = a^2b^2 : b^2c^2 : c^2a^2,$$

are called the Brocard points. The angles OCA, OAB, OBC, $O'BA$, $O'CB$, $O'AC$ are called the Brocard angles.

1. Show that the Brocard angles are all equal each to

$$\cot^{-1}[\cot A + \cot B + \cot C].$$

2. Find the equation to the Brocard line OO'.

3. Find the equation to the Brocard circle through O, O' and the centre of the circumscribed circle.

4. Given the base BC and the Brocard angle of a triangle, find the locus of the vertex.

5. Show that the bisectrices of the angles of ABC bisect the angles between the medians and the "symmedian lines"

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c}.$$

6. Show that the Brocard circle contains the symmedian point.

SOLUTION.

1. Let $OCA = \omega$. Then the equation of the line CO is

$$\frac{\alpha}{\beta} = \frac{\sin(C - \omega)}{\sin \omega} = \sin C(\cot \omega - \cot C).$$

But by the given equations,

$$\frac{\alpha}{\beta} = \frac{c^2}{ab} = \frac{\sin^2 C}{\sin A \sin B} = \sin C(\cot A + \cot B).$$

$$\therefore \quad \cot \omega - \cot C = \cot A + \cot B,$$

or

$$\omega = \cot^{-1}(\cot A + \cot B + \cot C).$$

* Prof. Barton also sent a correct solution of 98.

Similar reasoning gives the same value for each of the Brocard angles.

2. O is the intersection of the lines,

$$ab\alpha - c^2\beta = 0, \quad bc\beta - a^2\gamma = 0. \qquad (1)$$

The equation of any line passing through O is

$$m_1ab\alpha + (m_2bc - m_1c^2)\beta - m_2a^2\gamma = 0.$$

So the equation of any line passing through O' is

$$m_3c^2\alpha + (m_4a^2 - m_3ab)\beta - m_4bc\gamma = 0.$$

For the line OO' these equations must be identical. Hence by elimination and division we get

$$\frac{\alpha}{a}(a^4 - b^2c^2) + \frac{\beta}{b}(b^4 - a^2c^2) + \frac{\gamma}{c}(c^4 - a^2b^2) = 0.$$

3. As all circles intersect in the circular points at infinity, a circle is expressed by the equation

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma) = 0.$$

Substituting the values of α and γ given by equation (1), we get

$$lac^2 + mba^2 + ncb^2 + abc = 0 \qquad (2)$$

as the condition that the circle passes through O. In like manner if it passes through O',

$$lab^2 + mbc^2 + nca^2 + abc = 0. \qquad (3)$$

The centre of the circumscribed circle is the intersection of the two lines,

$$\alpha \cos B = \beta \cos A, \quad \beta \cos C = \gamma \cos B.$$

Substituting these values of α and β in the general equation of the circle, and reducing by the formulæ,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

we get

$$l(a^3 - ab^2 - ac^2) + m(b^3 - bc^2 - ba^2) + n(c^3 - ca^2 - cb^2) - abc = 0. \qquad (4)$$

Eliminating between (2), (3), and (4), we get

$$l = -\frac{bc}{a^2 + b^2 + c^2}, \quad m = -\frac{ac}{a^2 + b^2 + c^2}, \quad n = -\frac{ab}{a^2 + b^2 + c^2}.$$

This makes the equation of the required circle

$$abc(\alpha^2 + \beta^2 + \gamma^2) - c^3\alpha\beta - b^3\gamma\alpha - a^3\beta\gamma = 0. \qquad (5)$$

4. To express the locus of the point A in rectangular co-ordinates, make CB the axis of x, and C the origin. Then the point A is given by the equation,

$$\cot C = \frac{x}{y}, \quad \cot B = \frac{a - x}{y}, \quad \cot A = \frac{1 - \cot B \cot C}{\cot B + \cot C},$$

$$\cot \omega = \cot A + \cot B + \cot C,$$

as shown above. If ω and a are given, the resultant of these equations is the equation to the circle,

$$ay \cot \omega = a^2 - ax + x^2 + y^2.$$

5. If in the equation $\alpha / a = \beta / b$ we interchange α and β, we get the equation of the median line $\beta / a = \alpha / b$. But this interchange is equivalent to rotating the system about the bisectrix of C. Hence the bisectrices of the angles of ABC bisect the angles between the medians and the symmedians. The proposition as originally stated is incorrect.

6. In equation (5) put a, b, c for α, β, γ, and the equation is satisfied. Hence the Brocard circle (5) contains the symmedian point,

$$\alpha : \beta : \gamma = a : b : c.$$ [*C. B. Seymour.*]

104

A SPIRAL of Archimedes having its pole in the circumference of a given circle passes through the centre and the point 120° of the circumference from the pole. Find its equation. [*O. Root, Jr.*]

SOLUTION.

Let α be the angle made by the diameter through the pole with the initial axis. Then the curve

$$\rho = a\theta$$

passes through the centre, so that

$$R = a\alpha,$$

and also through the given point, so that

$$R(\sqrt{3} - 1) = (2n - \tfrac{1}{6})\,a\pi.$$

Hence $$\alpha = \frac{R}{a}, \quad \frac{a}{R} = \frac{\sqrt{3} - 1}{(2n - \frac{1}{6})\,\pi}.$$

[*R. H. Graves; Henry Heaton.*]

105

IF the centre of the circumscribed circle of a triangle is on the circumference of the inscribed circle,

$$\cos \tfrac{1}{2}A \,.\, \cos^{\frac{1}{2}} A + \cos \tfrac{1}{2}B \,.\, \cos^{\frac{1}{2}} B + \cos \tfrac{1}{2}C \,.\, \cos^{\frac{1}{2}} C = 0.$$

If the same point is on the circumference of one of the escribed circles, a like relation holds. [*W. M. Thornton.*]

SOLUTION.

The equation to the inscribed circle is, in trilinears,

$$\cos \tfrac{1}{2}A \,.\, \alpha^{\frac{1}{2}} + \cos \tfrac{1}{2}B \,.\, \beta^{\frac{1}{2}} + \cos \tfrac{1}{2}C \,.\, \gamma^{\frac{1}{2}} = 0.$$

The co-ordinates of the centre of the circumscribed circle satisfy the equations,

$$\frac{a}{\cos A} = \frac{\beta}{\cos B} = \frac{\gamma}{\cos C}.$$

Hence the required relation follows.

Treat, in a similar manner the equations to the escribed circles, viz:—

$$\cos \tfrac{1}{2}A \,.\, (-a)^{\frac{1}{2}} + \sin \tfrac{1}{2}B \,.\, \beta^{\frac{1}{2}} + \sin \tfrac{1}{2}C \,.\, \gamma^{\frac{1}{2}} = 0, \text{ etc.}$$

[*R. H. Graves.*]

106

If m be a positive integer,

$$\sin(m-1)\varphi - x\sin m\varphi + x^m \sin\varphi$$

will contain $1 - 2x\cos\varphi + x^2$. [*A. Hall.*]

SOLUTION.

The remainder, after a division of

$$R_n = x^n \sin(m-n-1)\varphi - x^{n+1}\sin(m-n)\varphi$$

by $1 - 2x\cos\varphi + x^2$, is by actual calculation found to be

$$R_{n+1} = x^{n+1}\sin(m-n-2)\varphi - x^{n+2}\sin(m-n-1)\varphi.$$

Accordingly, the division of

$$\sin(m-1)\varphi - x\sin m\varphi,$$

by $1 - 2x\cos\varphi + x^2$, will give the successive remainders,

$$R_1, R_2, R_3, \ldots R_m,$$

where $R_m = x^m \sin(-\varphi)$.

This proves the proposition. [*T. U. Taylor.*]

[Solved also by Professors Hall, Thornton, and Stone.]

EXERCISES.

122

If $P_1P_2P_3$ be a triangle inscribed in an ellipse, the co-ordinates of the point of concourse of its altitudes are given by the relations,

$$2ax = (a^2+b^2)(\cos p_1 + \cos p_2 + \cos p_3) - (a^2-b^2)\cos(p_1+p_2+p_3),$$
$$2by = (a^2+b^2)(\sin p_1 + \sin p_2 + \sin p_3) - (a^2-b^2)\sin(p_1+p_2+p_3);$$

where p_1, p_2, p_3 are the eccentric anomalies of the vertices. Prove these relations,

and find the values of x, y when for a given P the area of $P_1P_2P_3$ is greatest. Show that the locus of the orthocentre of all such triangles is also an ellipse.
[*W. M. Thornton.*]

123

FIND an expression for the area of a quadrilateral inscribed in an ellipse in terms of the eccentric anomalies of its vertices and the axes of the curve.
[*R. H. Graves.*]

124

THE area of the curve

$$r^2 = \frac{(a^2 - b^2)^2 \sin^2\theta \cos^2\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta}$$

is one-half of that of its circumscribing circle. [*R. H. Graves.*]

125

IF M be an integer prime to N, show that the number of places in the period of the repetend M/N is a divisor of

$$\varphi(N) = N\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\ldots\left(1 - \frac{1}{h}\right),$$

where $a, b, \ldots h$ are the prime factors of N and $\varphi(N)$ is the number of integers less than N and prime to N. [*Wm. E Heal.*]

126

FROM the centre of each of two equal coins a coin is cut at random. If one of these random coins be placed on the other at random in a horizontal position, what is the probability that the top coin will not fall off, supposing one coin just as likely to be placed on top as the other? [*Artemas Martin.*]

127

A VESSEL of depth a, the top and bottom of which are horizontal planes, is filled with a transparent fluid, the refractive index of which at a depth z below the surface is $1 + z/a$. Two small holes being made in the top, a ray of light enters at one hole, is reflected at the bottom, and emerges at the second hole; show that the distance between the holes must not be greater than

$$2a \log_e (2 + \sqrt{3}).$$ [*William Hoover.*]

YALE SENIOR PRIZE PROBLEMS.

128

A RHUMB line which cuts the meridians at the angle of 30° is projected upon a plane tangent to the south pole. The centre of projection is: 1° at the centre of the sphere; 2° at the north pole; 3° at an infinite distance on the line of the axis. Required the equation of the projection of the curve in each case.

129

THE two equal sides of an isosceles triangle move on two fixed points. Required the equation of the locus of the centre of gravity of the triangle.

130

THERE are three concentric circles of radii a, b, and c. In the circumference of each a point is taken at random. What is the average value of the square of the area of the triangle having its vertices at these points.

131

FIND the most general equation of the curve of the fourth degree which shall consist of two equal symmetrical loops and have no other branch, thus forming approximately a figure eight. Discuss the changes which result from causing the constants to vary.

132

A BROKEN line, $ABCDE$ . . . etc., is drawn in a plane, having all its angles equal and the concavity always on the same side. Each of the successive parts BC, CD, DE, etc., is half as long as the preceding. The length and direction of AB are given and the common angle. Required the direction and distance from A of the point to which the end of the line approaches, when the construction as described is continued indefinitely.

133

A, B, C, D are four points in a plane. The inverse points (with regard to a circle in the same plane) are a, b, c, d. What relations, independent of position, exist between the sides and angles of the quadrilaterals $ABCD$ and $abcd$.

134

LET ρ_a, ρ_b, ρ_c, ρ_d, ρ_e be five vectors drawn from a commoh but undetermined origin to five given points A, B, C, D, E. Find five scalars a, b, c, d, e such that the vector equation

$$a\rho_a + b\rho_b + c\rho_d + d\rho_d + e\rho_e = 0$$

shall hold true for all positions of the origin.

135

A UNIFORM elastic cord weighs 100 ounces. Its length when unstretched is 100 feet. If it is laid on a smooth table and stretched, its length increases one foot for every ounce of force applied. It is thrown over two pullies twenty feet apart, and on the same level, so that the central loop balances the pendent ends. Discuss the curve and the tensions.

136

DEVELOP a method of determining solar parallax by observations of the minor planets.

CONTENTS.

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